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NOTE

A Note on Pointwise Best Approximation

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The aim of this note is to fill in a gap in our previous paper in this journal. Precisely, we give a new proof of the following theorem: let $(\Omega, \mathscr{A}, \mu)$ be a σ-finite measure space with $\mu(\Omega) > 0$, $0 < p < +\infty$, and Y a separable subspace of a Banach space X. Then Y is proximinal in X iff $L^p(\mu, Y)$ is proximinal in $L^p(\mu, X)$.

1. INTRODUCTION

Because of our negligence and misuse of Theorem 1′ of [1, p. 230], there is a gap in the process of the proof of Theorem 3.4 in our previous paper [2]. However, Theorem 3.4 of [2] itself is correct. Along the idea of the proof of Theorem 3.4 used in [2], we only need to slightly modify the process of the proof of Theorem 3.4 of [2] to complete its proof by making use of Theorem 5.10 of [3] instead of Theorem 1′ of [1].

Let $(\Omega, \mathscr{A}, \mu)$ be a measure space, $(B, \|\cdot\|)$ a Banach space, and $(\Omega, \hat{\mathscr{A}}, \hat{\mu})$ the Lebesgue extension of $(\Omega, \mathscr{A}, \mu)$ (for the notation of a Lebesgue extension, see [4, pp. 142–143]). Ordinarily, a B-valued strongly measurable function on $(\Omega, \mathscr{A}, \mu)$ means it is the limit almost everywhere of a sequence of $\mathscr{A}$-measurable simple B-valued functions on $(\Omega, \mathscr{A}, \mu)$

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[5]. One can easily see that the notation of strongly measurable functions coincides with that of μ-measurable functions of [4] when $(\Omega, \mathscr{A}, \mu)$ is σ-finite.

In [2], to avoid the difficulty of discussing measurability, we introduced the concept of strongly $\mathscr{A}$-measurable functions. A so-called B-valued strongly $\mathscr{A}$-measurable function V on $(\Omega, \mathscr{A}, \mu)$ is the pointwise limit of a sequence $\{\varphi_n\}$ of $\mathscr{A}$-measurable simple B-valued functions on Ω, namely, $\|V(\omega) - \varphi_n(\omega)\| \to 0$ for any $\omega \in \Omega$. Clearly, V is strongly $\mathscr{A}$-measurable iff V is $\mathscr{A}$-measurable and the range of V is a separable subset of B [6, p. 48]. The concept of strongly $\mathscr{A}$-measurable functions on $(\Omega, \mathscr{A}, \mu)$ depends only on the measurable space $(\Omega, \mathscr{A})$ and is independent of the measure μ.

Lemma III 6.9 of [4, p. 147] shows that when $(\Omega, \mathscr{A}, \mu)$ is σ-finite, $V: (\Omega, \mathscr{A}, \mu) \to B$ is μ-measurable iff V is $\hat{\mathscr{A}}$-measurable and μ-essentially separably valued, and hence also iff V is μ-equivalent to a strongly $\hat{\mathscr{A}}$-measurable function (and thus μ-equivalent to a strongly $\mathscr{A}$-measurable function).

Throughout this note, let $(\Omega, \mathscr{A}, \mu)$ be a σ-finite measure space with $\mu(\Omega) > 0$, $(X, \|\cdot\|)$ be a Banach space. Denote by $L^0(\mathscr{A}, X)$ the linear space of all X-valued strongly $\mathscr{A}$-measurable functions on $(\Omega, \mathscr{A}, \mu)$; by $(\Omega, \hat{\mathscr{A}}, \hat{\mu})$ the Lebesgue extension of $(\Omega, \mathscr{A}, \mu)$ (and hence $(\Omega, \hat{\mathscr{A}}, \hat{\mu})$ is a complete σ-finite measure space); by $L^0(\hat{\mathscr{A}}, X)$ the linear space of all X-valued strongly $\hat{\mathscr{A}}$-measurable functions on Ω; by $M^0(\mu, X)$ the linear space of all X-valued μ-measurable functions on Ω; by $L(\mathscr{A}, X)$ (resp., $L(\hat{\mathscr{A}}, X)$) the linear space of all the μ-equivalence classes of elements in $L^0(\mathscr{A}, X)$ (resp., $L^0(\hat{\mathscr{A}}, X)$); and by $M(\mu, X)$ the linear space of all the μ-equivalence classes of elements in $M^0(\mu, X)$.

2. MAIN RESULTS AND THEIR PROOFS

First, let us note that although $L^0(\mathscr{A}, X)$, $L^0(\hat{\mathscr{A}}, X)$, and $M^0(\mu, X)$ may be, essentially, different spaces from each other (when X is separable, $M^0(\mu, X) = L^0(\hat{\mathscr{A}}, X)$), however, as spaces of μ-equivalence classes of the related functions, $L(\mathscr{A}, X)$, $L(\hat{\mathscr{A}}, X)$, and $M(\mu, X)$ can be essentially identified.

Theorem 1. *Let Y be a separable and proximinal subspace of X. Then $L(\mathscr{A}, Y)$ is pointwise proximinal in $L(\mathscr{A}, X)$.*

Proof. For any fixed element p in $L(\mathscr{A}, X)$, let $p^0 \in L^0(\mathscr{A}, X)$ be any selected representative of p. Define a mapping $f: \Omega \times Y \to R^1$ by $f(\omega, y) = \|p^0(\omega) - y\| - \text{dist}(p^0(\omega), Y)$ for any $(\omega, y) \in \Omega \times Y$. Then f is a Carathéodory-type mapping, i.e., $f(\cdot, y)$ is $\mathscr{A}$-measurable for each $y \in Y$, and $f(\omega, \cdot)$ is continuous for each $\omega \in \Omega$. Thus by Lemma 7.5 of [3, p. 877], f is $\mathscr{A} \otimes \mathscr{B}(Y)$-measurable; of course, f is also $\hat{\mathscr{A}} \otimes \mathscr{B}(Y)$-measurable.

Now define a multifunction $F: \Omega \to 2^{\Omega}$ by $F(\omega) = \{ y \in Y \mid \|p^0(\omega) - y\| = \operatorname{dist}(p^0(\omega), Y)\}$ for any $\omega \in \Omega$. Since Y is separable and proximinal, Y is closed, and hence also a complete separable Banach space. $\mathrm{Gr}F = \{(\omega, y) \in \Omega \times Y \mid y \in F(\omega)\} = f^{-1}(\{0\}) \in \mathscr{A} \otimes \mathscr{B}(Y) \subset \hat{\mathscr{A}} \otimes \mathscr{B}(Y)$, since $\hat{\mathscr{A}}$ is a Suslin family and Y is a Suslin space, and by Theorem 5.10 of [3, p. 873], F has a Castaing representation. F is closed, and thus there exists a countable set $\{p_n^0 \mid n \in N\}$ in $L^0(\hat{\mathscr{A}}, Y)$ such that $\overline{\{p_n^0(\omega) \mid n \in N\}} = F(\omega)$ for each $\omega \in \Omega$. This shows $\operatorname{dist}(p^0(\omega), Y) = \|p^0(\omega) - p_n^0(\omega)\|$ for any $n \in N$ and any $\omega \in \Omega$. Since each p_n^0 must be μ-equivalent to an element q_n^0 in $L^0(\mathscr{A}, Y)$, we also have $\operatorname{dist}(p^0(\omega), Y) = \|p^0(\omega) - q_n^0(\omega)\|$ a.e. for each $n \in N$, and hence $\|p^0(\omega) - q_n^0(\omega)\| \leqslant \|p^0(\omega) - q^0(\omega)\|$ a.e. for each $q^0 \in L^0(\mathscr{A}, Y)$ and each $n \in N$. Let q_n be the μ-equivalence class of q_n^0 for each $n \in N$. Then $\{q_n\} \subset L(\mathscr{A}, Y)$ and $\|p - q_n\| \leqslant \|p - q\|$ for any $n \in N$ and any $q \in L(\mathscr{A}, Y)$, which shows each q_n is just a pointwise best approximant of p in $L(\mathscr{A}, Y)$.

This completes the proof of Theorem 1.

Remark 1. In [2], since functions that are equal a.e. are identified, take $(\Omega, \mathscr{A}, \mu) = (S, \Sigma, \mu)$ of [2] there. Then $L(S, X)$ of [2] is just $L(\mathscr{A}, X)$ here, and hence Theorem 1 here is just Theorem 3.4 of [2].

COROLLARY 1 [2, Theorem 3.5, p. 319]. *Let Y be a separable subspace of X, and p be a positive real number. Then $L^p(\mu, Y)$ is proximinal in $L^p(\mu, X)$ iff Y is proximinal in X.*

Proof. This follows immediately from Theorem 1 here and Theorems 3.2 and 3.3 of [2].

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